Domain Wall and Periodic Solutions of Coupled ϕ^6 and Coupled ϕ^6-ϕ^4 Models

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Abstract:

We obtain several higher order periodic solutions of a Coupled ϕ^6 model in terms of Lamé polynomials of order one and two. These solutions are unusual in the sense that while they are the solutions of the coupled problem, they are not the solutions of the uncoupled problem. We also obtain exact solutions of coupled ϕ^6-ϕ^4 models, both when the ϕ^4 potential corresponds to a first order (asymmetric double well) or a second order (symmetric double well) transition.

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1   Introduction

Coupled triple well or ϕ^6 models [1, 2] arise in the context of many first order structural phase transitions. There exist analogous coupled models in field theoretical contexts [3, 4, 5]. Specifically, when a first order transition is driven by two primary order parameters, the free energy should be expanded to sixth order in both order parameters with a bi-quadratic (or possibly other symmetry allowed) coupling. In a recent publication we obtained a large number of periodic solutions of a coupled ϕ^6 model with bi-quadratic coupling [6]. All these solutions had the feature that in the uncoupled limit, they reduce to the well known solutions of the corresponding uncoupled ϕ^6 problem. The purpose of this paper is to point out that this coupled model has truly novel solutions in terms of Lamé polynomials of order one and two [7, 8], provided we add (symmetry allowed) quartic-quadratic and quadratic-quartic couplings to the coupled ϕ^6 model considered earlier [6]. It may be noted that these solutions exist only because of the coupling between the two fields (up to sixth order). In other words, while the Lamé polynomials of order one and two are the solutions of the coupled problem, they are not the solutions of the uncoupled problem.

In this context, we note that in recent publications we have obtained a large number of periodic solutions, in terms of Lamé polynomials of order one and two, of coupled ϕ^4 problems, both when the ϕ^4 potential corresponds to the second as well as to the first order transition [9, 10, 11]. The obvious question then is whether one can also obtain exact solutions of the coupled ϕ^6-ϕ^4 problems, both when the ϕ^4 potential corresponds to the second as well as to the first order transition. Interestingly, examples of these situations occur in condensed matter. The face-centered cubic to monoclinic transition in Pu involves an intermediate phase hexagonal to monoclinic transition with a free energy modeled by the coupled ϕ^6-asymmetric ϕ^4 model [12]. Similarly, the coexistence of face-centered cubic, body-centered cubic and hexagonal close-packed structures in cobalt [13] as well as Fe and Tl [14] is modeled by the same free energy. An example of coupled ϕ^6-symmetric ϕ^4 model with a bi-quadratic coupling is the triggered ferroelectric transition [15, 16]. Thus, another important purpose of this paper is to obtain exact solutions of the coupled ϕ^6-ϕ^4 models with bi-quadratic coupling, both when the ϕ^4 potential corresponds to the first as well as to the second order transition.

The plan of the paper is the following. In Sec. II we provide novel periodic as well as the corresponding

hyperbolic solutions in terms of Lamé polynomials of *order one* for the coupled ϕ^6 model with an explicit bi-quadratic as well as quadratic-quartic and quartic-quadratic couplings. In Sec. III we provide novel periodic as well as the corresponding hyperbolic solutions in terms of Lamé polynomials of *order two* for the same coupled ϕ^6 model. In Sec. IV we provide the exact solutions of the coupled $\phi^6 - \phi^4$ problem with bi-quadratic coupling in case the ϕ^4 potential corresponds to either the first or the second order transition. Finally, in Sec. V we conclude with summary and possible extensions.

2 The Coupled ϕ^6 Model and Solutions in Terms of Lamé Polynomials of Order One

In [6] we had considered the following coupled ϕ^6 model, with a bi-quadratic coupling, in one dimension with the potential

$$V(\phi, \psi) = \left(\frac{a_1}{2} \phi^2 - \frac{b_1}{4} \phi^4 + \frac{c_1}{6} \phi^6 \right) + \left(\frac{a_2}{2} \psi^2 - \frac{b_2}{4} \psi^4 + \frac{c_2}{6} \psi^6 \right) + \frac{d}{2} \phi^2 \psi^2 . \tag{1}$$

We now show that in case we add the following quartic-quadratic and quadratic-quartic coupling terms

$$V' = \frac{e}{4} \phi^4 \psi^2 + \frac{f}{2} \phi^2 \psi^4 , \tag{2}$$

to the potential (1), then in addition to the solutions obtained in [6], there exist truly novel solutions in terms of Lamé polynomials of order one and two to this coupled problem. Here $a_{1,2}, b_{1,2}, c_{1,2}, d, e$ and f are material (or system) dependentparameters; ϕ and ψ are scalar fields. From stability considerations we shall always take $c_1, c_2 > 0$. Further, since we are interested in a model for first order transition, we shall take $b_1, b_2 > 0$. As far as a_1, a_2 are concerned, their sign is arbitrary and the shape of the potential depends on the ratio $b_1^2/4a_1c_1$ and $b_2^2/4a_2c_2$. In particular, in the decoupled limit (i.e. $d = e = f = 0$), it is easily shown that as long as $4a_1c_1 > b_1^2$, the potential has a minimum at $\phi = 0$ [17, 18]. In the case $4a_1c_1 = b_1^2$, apart from the minimum at $\phi = 0$ one now has points of inflection at $\phi^2 = b_1/2c_1$. As a_1 decreases further so that $4a_1c_1 < b_1^2 < (16/3)a_1c_1$, one finds that while $\phi = 0$ is still the absolute minimum, one now has two local minima and two maxima at

$$\phi_{min}^2 = \frac{b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2c_1} , \quad \phi_{max}^2 = \frac{b_1 - \sqrt{b_1^2 - 4a_1c_1}}{2c_1} . \tag{3}$$

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At the special value of $(16/3)a_1c_1 = b_1^2$ one has three degenerate minima at $\phi = 0$ and at ϕ_{min}^2 as given by Eq. (3); for relevant figures, see [17, 18]. This is the point of first order transition. As a_1 decreases further so that $0 < (16/3)a_1c_1 < b_1^2$, then the roles of the minima are reversed, now $\phi = 0$ is the local minimum while ϕ_{min}^2 as given by Eq. (3) are the two degenerate absolute minima and ϕ_{max}^2 are the two maxima. Finally as $a_1 \leq 0$, the potential has two absolute minima at ϕ_{min}^2 as given by Eq. (3) while $\phi = 0$ is now the sole maximum. This picture continues to persist, for arbitrarily large and negative a_1. Throughout this paper, we shall term the point where $b_1^2 = (16/3)a_1c_1$ as the point of first order transition, i.e. the point with $T = T_c$. On the other hand, the point where local structure (i.e. local minima) starts growing with decreasing temperature, i.e. $b_1^2 = 4a_1c_1$, will be termed as $T = T_p$. Note that for $T > T_p$ there is only a global minimum at $\phi = 0$ and there are no other extrema. Thus the region where $b_1^2 > (16/3)a_1c_1$ corresponds to $T < T_c$ while the region with $4a_1c_1 < b_1^2 < (16/3)a_1c_1$ corresponds to $T_c < T < T_p$. A similar analysis is also true for the potential in ψ with a_1, b_1, c_1 being replaced by a_2, b_2, c_2, respectively.

The (static) equations of motion which follow from Eqs. (1) and (2) are

$$\frac{d^2\phi}{dx^2} = a_1\phi - b_1\phi^3 + c_1\phi^5 + d\phi\psi^2 + e\phi^3\psi^2 + f\phi\psi^4 \,,$$
$$\frac{d^2\psi}{dx^2} = a_2\psi - b_2\psi^3 + c_2\psi^5 + d\psi\phi^2 + \frac{e}{2}\phi^4\psi + f\phi^2\psi^3 \,. \tag{4}$$

These coupled equations have thirteen distinct periodic (elliptic function) solutions, i.e. five "bright-bright", three "bright-dark" and five "dark-dark" solutions which have already been discussed in [6] (in case $e = f = 0$). In particular, there are five solutions below the transition temperature T_c, four at T_c, one above T_c (i.e. $T_c < T < T_p$), and three in the mixed phase in the sense that while one of the field is above T_c, the other one is below T_c. The latter situation is akin to the one found in multiferroic materials where one transition (i.e. antiferromagnetic) takes place at a higher temperature than the other transition (e.g. ferroelectric) or vice versa [19]. It is worth pointing out that in turn in the single soliton limit, these lead to *eight* distinct coupled (hyperbolic) soliton solutions. In particular, one obtains three solutions below the transition temperature T_c, two at T_c, one above T_c and two in the mixed phase in the sense that while one of the field is above T_c, the other one is below T_c.

We now show that apart from these solutions, we also have rather unusual solutions in terms of Lamé polynomials of order one (and two discussed in the next section) which we now discuss one by one. Since

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there are three Lamé polynomials of order one (i.e. sn, cn, dn) and since the field equations are essentially symmetric in ϕ and ψ, we expect six independent solutions to the coupled field equations in terms of Lamé polynomials of order one. In particular, we first show that there are three periodic bright-bright, two periodic dark-bright and one periodic dark-dark soliton solutions in terms of Lamé polynomials of order one, which in turn lead to one bright-bright, one dark-dark and one dark-bright hyperbolic soliton solution.

2.1 Solution I

We look for the most general solutions to the coupled Eqs. (4) in terms of the Jacobi elliptic functions $sn(x, m)$, $cn(x, m)$ and $dn(x, m)$ [7] where the modulus $m \equiv k^2$. It is easily shown that

$$\phi = A\mathrm{sn}(Dx + x_0, m), \quad \psi = B\mathrm{sn}(Dx + x_0, m), \tag{5}$$

is an exact dark-dark periodic solution to the coupled Eqs. (4) provided the following six coupled equations are satisfied

$$a_1 = -(1+m)D^2, \tag{6}$$

$$-b_1 A^2 + dB^2 = 2mD^2, \tag{7}$$

$$c_1^4 + eA^2 B^2 + \frac{f}{2}B^4 = 0, \tag{8}$$

$$a_2 = -(1+m)D^2, \tag{9}$$

$$-b_2 B^2 + dA^2 = 2mD^2, \tag{10}$$

$$c_2 B^4 + fA^2 B^2 + \frac{e}{2}A^4 = 0. \tag{11}$$

Here, A and B denote the amplitudes of the kink lattice, D is an inverse characteristic length while x_0 is the (arbitrary) location of the kink. Three of these equations determine the three unknowns A, B, D while the other three equations give three constraints between the nine parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e, f$. In particular, we find that the solution exists only if $a_1 < 0, a_2 < 0, e < 0, f < 0$. We obtain

$$D^2 = \frac{|a_1|}{(1+m)}, \quad A^2 = \frac{2mD^2(d+b_2)}{d^2 - b_1 b_2}, \quad B^2 = \frac{(d+b_1)A^2}{(d+b_2)}, \tag{12}$$

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while the three constraints are

$$a_1 = a_2 < 0, \quad (4c_1c_2 - |e||f|)(d + b_1) = 2(e^2 + 2|f|c_1)(d + b_2),$$

$$(4c_1c_2 - |e||f|)^2 = 4(e^2 + 2|f|c_1)(f^2 + 2|e|c_2). \tag{13}$$

In the limit of $m = 1$, the periodic solution (5) goes over to the hyperbolic dark-dark soliton solution

$$\phi = A\tanh(Dx + x_0), \quad \psi = B\tanh(Dx + x_0), \tag{14}$$

provided the constraints (12) and (13) with $m = 1$ are satisfied.

2.2 Solution II

It is easy to show that

$$\phi = A\mathrm{cn}(Dx + x_0, m), \quad \psi = B\mathrm{cn}(Dx + x_0, m), \tag{15}$$

is an exact bright-bright periodic solution to the coupled Eqs. (4) provided six coupled equations similar to Eqs. (6) to (11) are satisfied. Three of these equations again determine the three unknowns A, B, D while the other three equations give three constraints between the nine parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e, f$. In particular, we obtain

$$D^2 = \frac{a_1}{(2m - 1)}, \quad a_1 = a_2, \quad A^2 = \frac{2m(d + b_2)D^2}{(b_1 b_2 - d^2)}, \quad B^2 = \frac{(b_1 + d)A^2}{(b_2 + d)}, \tag{16}$$

while the remaining two constraints are again given by Eq. (13). Note that $a_1 = a_2 > (<)0$ if $m > (<)1/2$.

In the limit of $m = 1$, the periodic solution (15) goes over to the hyperbolic bright-bright solution

$$\phi = A\mathrm{sech}(Dx + x_0), \quad \psi = B\mathrm{sech}(Dx + x_0), \tag{17}$$

provided the constraints (13) and (16) with $m = 1$ are satisfied.

2.3 Solution III

Yet another bright-bright periodic soliton solution is

$$\phi = A\mathrm{dn}(Dx + x_0, m), \quad \psi = B\mathrm{dn}(Dx + x_0, m), \tag{18}$$

provided six coupled equations similar to Eqs. (6) to (11) are satisfied. Three of these equations determine the three unknowns A, B, D while the other three equations give three constraints between the nine parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e, f$. In particular, we obtain

$$D^2 = \frac{a_1}{(2-m)}\,, \quad a_1 = a_2 > 0\,, \quad A^2 = \frac{2(d+b_2)D^2}{(b_1 b_2 - d^2)}\,, \quad B^2 = \frac{(b_1+d)A^2}{(b_2+d)}\,, \tag{19}$$

while the remaining two constraints are again given by Eq. (13).

In the limit of $m = 1$, the periodic solution (18) again goes over to the hyperbolic bright-bright soliton solution (17).

Note that while for the solution (5), $d^2 > b_1 b_2$, for the solutions (15) and (18), its the other way around, i.e. $d^2 < b_1 b_2$.

2.4 Solution IV

Yet another bright-bright periodic soliton solution is

$$\phi = A\sqrt{m}\,\mathrm{cn}(Dx + x_0, m)\,, \quad \psi = B\,\mathrm{dn}(Dx + x_0, m)\,, \tag{20}$$

provided the following six coupled equations are satisfied

$$a_1 + \frac{f}{2}(1-m)^2 B^4 + d(1-m)B^2 = (2m-1)D^2\,, \tag{21}$$

$$b_1 A^2 + dB^2 - (1-m)eA^2 B^2 - (1-m)fB^2 = 2D^2\,, \tag{22}$$

$$c_1 A^4 + eA^2 B^2 + \frac{f}{2}B^4 = 0\,, \tag{23}$$

$$a_2 + \frac{e}{2}(1-m)^2 A^4 - (1-m)dA^2 = (2-m)D^2\,, \tag{24}$$

$$b_2 B^2 + dA^2 + (1-m)fA^2 B^2 + (1-m)eA^4 = 2D^2\,, \tag{25}$$

$$c_2 B^4 + fA^2 B^2 + \frac{e}{2}A^4 = 0\,. \tag{26}$$

Three of these equations determine the three unknowns A, B, D while the other three equations give three constraints between the nine parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e, f$. In particular, this solution is also valid only if $e < 0, f < 0$. Further, two of the relations are given by

$$(4c_1 c_2 - |e||f|)^2 = 4(e^2 + 2|f|c_1)(f^2 + 2|e|c_2)\,, \quad (4c_1 c_2 - |e||f|)A^2 = 2(f^2 + 2|e|c_2)B^2\,. \tag{27}$$

In the limit of $m = 1$, the periodic solution (19) again goes over to the hyperbolic bright-bright soliton solution (17).

2.5 Solution V

In addition there are two dark-bright periodic soliton solutions. One of them is

$$\phi = A\mathrm{sn}(Dx + x_0, m), \quad \psi = B\mathrm{cn}(Dx + x_0, m), \tag{28}$$

provided the following six coupled equations are satisfied

$$a_1 + \frac{f}{2}B^4 + dB^2 = -(1 + m)D^2, \tag{29}$$

$$-b_1 A^2 - dB^2 + eA^2 B^2 - fB^4 = 2mD^2, \tag{30}$$

$$c_1 A^4 + \frac{f}{2}B^4 = eA^2 B^2, \tag{31}$$

$$a_2 + \frac{e}{2}A^4 + dA^2 = (2m - 1)D^2, \tag{32}$$

$$b_2 B^2 + dA^2 - fA^2 B^2 + eA^4 = 2mD^2, \tag{33}$$

$$c_2 B^4 + \frac{e}{2}A^4 = fA^2 B^2. \tag{34}$$

Three of these equations determine the three unknowns A, B, D while the other three equations give three constraints between the nine parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e, f$. For example, it is easily shown that

$$A^2 = \frac{b_1 \pm \sqrt{b_1^2 - 4a_1 c_1 - 4(1 - m)D^2 c_1}}{2c_1}, \quad B^2 = \frac{b_2 \pm \sqrt{b_2^2 - 4a_2 c_2 - 4D^2 c_2}}{2c_2}. \tag{35}$$

Further, unlike the previous four solutions, this solution exists only if $e > 0, f > 0$ and two of the constraints are given by

$$(4c_1 c_2 - ef)A^2 = 2(2ec_2 - f^2)B^2, \quad (4c_1 c_2 - ef)^2 = 4(2ec_2 - f^2)(2fc_1 - e^2), \tag{36}$$

while the other two constraints are

$$B^2 = \frac{b_1 A^2 - 2a_1 - 2D^2}{d + eA^2} = \frac{2a_2 + 2(1 - m)D^2 + dA^2}{b_2 - fA^2}. \tag{37}$$

In the limit of $m = 1$, the periodic solution (28) goes over to the hyperbolic dark-bright soliton solution

$$\phi = A\tanh(Dx + x_0), \quad \psi = B\mathrm{sech}(Dx + x_0), \tag{38}$$

satisfying the constraints (35) to (37) with $m = 1$.

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2.6 Solution VI

Another dark-bright periodic soliton solution is given by

$$\phi = A\sqrt{m}\,\text{sn}(Dx + x_0, m), \quad \psi = B\text{dn}(Dx + x_0, m), \tag{39}$$

provided six coupled equations similar to (29) to (34) are satisfied. Three of these equations determine the three unknowns A, B, D while the other three equations give three constraints between the nine parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e, f$. For example, it is easily shown that two of the constraints are again given by Eq. (36) while A^2 and B^2 are now given by

$$A^2 = \frac{b_1 \pm \sqrt{b_1^2 - 4a_1c_1 + 4(1-m)D^2c_1}}{2c_1}, \quad B^2 = \frac{b_2 \pm \sqrt{b_2^2 - 4a_2c_2 - 4mD^2c_2}}{2c_2}. \tag{40}$$

while the other two constraints are

$$B^2 = \frac{b_1A^2 - 2a_1 - 2mD^2}{d + eA^2} = \frac{2a_2 - 2(1-m)D^2 + dA^2}{b_2 - fA^2}. \tag{41}$$

In the limit of $m = 1$, the periodic solution (39) goes over to the hyperbolic dark-bright soliton solution (38).

3 Solutions of Coupled ϕ^6 Model In terms of Lamé Polynomials of Order Two

We now show that quite remarkably, the coupled model characterized by the field Eqs. (4) not only admits periodic solutions in terms of Lamé polynomials of order one, but it also admits novel periodic solutions in terms of Lamé polynomials of order two. It is worth reminding once again that neither Lamé polynomials of order one nor of order two are solutions of the uncoupled ϕ^6 problem. Since there are five Lamé polynomials of order two, and since two of these are of the form $A\text{sn}^2[D(x+x_0), m] + F$, and further, the two field Eqs. (4) are symmetrical in ϕ and ψ, in principle there could be ten solutions of order two. However, it turns out that only two of these are admitted by the field Eqs. (4) which we now discuss.

3.1 Solution I

It is easily shown that

$$\phi = A\text{sn}^2(Dx + x_0, m) + F, \quad \psi = B\text{sn}(Dx + x_0, m)\text{cn}(Dx + x_0, m), \tag{42}$$

is an exact periodic solution to the coupled Eqs. (4) provided the following eleven coupled equations are satisfied

$$a_1 F - b_1 F^3 + c_1 F^5 = 2AD^2, \tag{43}$$

$$a_1 A - 3b_1 AF^2 + 5c_1 AF^4 + eB^2 F^3 + dB^2 F = -4(1+m)AD^2, \tag{44}$$

$$-3b_1 A^2 F + 10c_1 A^2 F^3 + eB^2 F^2(3A - F) + \frac{f}{2}B^4 F + dB^2(A - F) = 6mAD^2, \tag{45}$$

$$-b_1 A^3 + 10c_1 A^3 F^2 + 3eAB^2 F(A - F) + \frac{f}{2}B^4(A - 2F) - dAB^2 = 0, \tag{46}$$

$$5c_1 A^4 F + eA^2 B^2(A - 3F) + \frac{f}{2}B^4(F - 2A) = 0, \tag{47}$$

$$c_1 A^4 - eA^2 B^2 + \frac{f}{2}B^4 = 0. \tag{48}$$

$$a_2 + \frac{e}{2}F^4 + dF^2 = -(4+m)D^2, \tag{49}$$

$$-b_2 B^2 + 2eAF^3 + fF^2 B^2 + 2dAF = 6mD^2, \tag{50}$$

$$b_2 B^2 + c_2 B^4 + 3eA^2 F^2 + fB^2 F(2A - F) + dA^2 = 0, \tag{51}$$

$$-2c_2 B^4 + 2eA^3 F + fAB^2(A - 2F) = 0, \tag{52}$$

$$c_2 B^4 - fA^2 B^2 + \frac{e}{2}A^4 = 0. \tag{53}$$

Four of these equations determine the four unknowns A, B, D, F while the other equations give constraints between the nine parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e, f$. In particular, we find that the solution exists only if

$$eA^2 = fB^2, \quad e^3 = 8c_1^2 c_2, \quad f^3 = 8c_2^2 c_1, \tag{54}$$

and further if $F \neq 0$.

In the limit of $m = 1$, the periodic solution (42) goes over to the hyperbolic solution

$$\phi = A\tanh^2(Dx + x_0) + F, \quad \psi = B\tanh(Dx + x_0)\text{sech}(Dx + x_0), \tag{55}$$

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provided the constraints (43) to (53) with $m = 1$ are satisfied. There is one special case when this solution takes a simpler form, i.e. when $A = -F$, the solution is given by

$$\phi = -A\text{sech}^2(Dx + x_0), \quad \psi = B\tanh(Dx + x_0)\text{sech}(Dx + x_0), \tag{56}$$

provided Eq. (54) is satisfied and further

$$D^2 = \frac{a_1}{4}, \quad a_1 = 4a_2 > 0, \quad d = -b_2, \quad b_2^2(f - e) = 6a_2c_2e,$$

$$B^2 = \frac{6a_2}{b_2}, \quad A^2 = \frac{-b_1 \pm \sqrt{b_1^2 - 6a_1c_1}}{2c_1}. \tag{57}$$

Thus in the ϕ variable, one is at $T < T_c^I$ since $b_1^2 > 6a_1c_1$.

3.2 Solution II

The other allowed solution is

$$\phi = A\text{sn}^2(Dx + x_0, m) + F, \quad \psi = B\text{sn}(Dx + x_0, m)\text{dn}(Dx + x_0, m), \tag{58}$$

which is an exact periodic solution to the coupled Eqs. (4) provided Eqs. (43), (44) and the following nine coupled equations are satisfied

$$-3b_1A^2F + 10c_1A^2F^3 + eB^2F^2(3A - mF) + \frac{f}{2}B^4F + dB^2(A - mF) = 6mAD^2, \tag{59}$$

$$-b_1A^3 + 10c_1A^3F^2 + 3eAB^2F(A - mF) + \frac{f}{2}B^4(A - 2mF) - dmAB^2 = 0, \tag{60}$$

$$5c_1A^4F + eA^2B^2(A - 3mF) + \frac{mf}{2}B^4(mF - 2A) = 0, \tag{61}$$

$$c_1A^4 - emA^2B^2 + \frac{f}{2}m^2B^4 = 0. \tag{62}$$

$$a_2 + \frac{e}{2}F^4 + dF^2 = -(1 + 4m)D^2, \tag{63}$$

$$-b_2B^2 + 2eAF^3 + fF^2B^2 + 2dAF = 6mD^2, \tag{64}$$

$$b_2mB^2 + c_2B^4 + 3eA^2F^2 + fB^2F(2A - mF) + dA^2 = 0, \tag{65}$$

$$-2mc_2B^4 + 2eA^3F + fAB^2(A - 2mF) = 0, \tag{66}$$

$$c_2m^2B^4 - fmA^2B^2 + \frac{e}{2}A^4 = 0. \tag{67}$$

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Four of these equations determine the four unknowns A, B, D, F while the other equations give constraints between the nine parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e, f$. In particular, we find that the solution exists only if

$$emA^2 = fB^2, \quad e^3 = 8c_1^2 c_2, \quad f^3 = 8c_2^2 c_1, \tag{68}$$

and further if $F \neq 0$.

In the limit of $m = 1$, the periodic solution (58) also goes over to the hyperbolic solution (55).

4   Coupled ϕ^6-ϕ^4 Model

We now consider a coupled ϕ^6-ϕ^4 model with bi-quadratic coupling. In particular, we consider the model characterized by the potential

$$V(\phi, \psi) = \left(\frac{a_1}{2}\phi^2 - \frac{b_1}{4}\phi^4 + \frac{c_1}{6}\phi^6 \right) + \left(\frac{a_2}{2}\psi^2 + \frac{f_2}{3}\psi^3 + \frac{b_2}{4}\psi^4 \right) + \frac{d}{2}\phi^2\psi^2. \tag{69}$$

This leads to the coupled field equations

$$\frac{d^2\phi}{dx^2} = a_1\phi - b_1\phi^3 + c_1\phi^5 + d\phi\psi^2,$$

$$\frac{d^2\psi}{dx^2} = a_2\psi + f_2\psi^2 + b_2\psi^3 + d\psi\phi^2. \tag{70}$$

From stability considerations we shall always take $c_1 > 0, b_2 > 0$. Note that in case $f_2 = 0$, the model corresponds to the symmetric ϕ^4 model with a second order transition, while as long as $f_2 \neq 0$, the model corresponds to a first order transition. We shall discuss the various solutions both when $f_2 \neq 0$ as well as when $f_2 = 0$.

4.1   Solution I

It is easily shown that

$$\phi = A\sqrt{1 \pm \operatorname{sn}(Dx + x_0, m)}, \quad \psi = B\operatorname{sn}(Dx + x_0, m) + F, \tag{71}$$

is an exact periodic solution to the coupled Eqs. (70) provided the following seven coupled equations are satisfied

$$a_1 - b_1 A^2 + c_1 A^4 + dF^2 = -\frac{D^2}{4}, \tag{72}$$

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$$-b_1 A^2 + 2c_1 A^4 \pm 2dBF = -\frac{mD^2}{2}, \tag{73}$$

$$c_1^4 + dB^2 = \frac{3mD^2}{4}, \tag{74}$$

$$[a_2 + f_2 F + b_2 F^2 + dA^2]F = 0, \tag{75}$$

$$a_2 B + 2f_2 FB + 3b_2 BF^2 + dA^2(B \pm F) = -(1+m)BD^2, \tag{76}$$

$$[f_2 B + 3b_2 BF \pm dA^2]B = 0, \tag{77}$$

$$b_2 B^2 = 2mD^2. \tag{78}$$

Here, A and B denote the amplitudes of the kink lattice, D is an inverse characteristic length, F is a constant while x_0 is the (arbitrary) location of the kink. Four of these equations determine the four unknowns A, B, D, F while the other three equations give three constraints between the seven parameters $a_{1,2}, b_{1,2}, c_1, f_2, d$.

In the limit of $m = 1$, the periodic solution (71) goes over to the hyperbolic solution

$$\phi = A\sqrt{1 \pm \tanh(Dx + x_0)}, \quad \psi = B \tanh(Dx + x_0) + F, \tag{79}$$

provided Eqs. (72) to (78) with $m = 1$ are satisfied.

Several comments are in order at this stage.

1. The solution (71) continues to exist even if $f_2 = 0$, i.e. in the symmetric ϕ^4 case.

2. Solution (71) continues to exist if $F = 0$. However, no solution exists in case $F = f_2 = 0$ as in that case d is also forced to be zero.

3. In case $B = \pm F$ then the solution exists only at $m = 1$. In particular it is easily shown that

$$\phi = A\sqrt{1 \pm \tanh(Dx + x_0)}, \quad \psi = F[1 \pm \tanh(Dx + x_0)], \tag{80}$$

is an exact solution to field Eqs. (70) provided

$$D^2 = a_1, \quad a_2 = 4a_1 > 0, \quad A^2 = \frac{2a_1}{b_1}, \quad B^2 = \frac{a_2}{2b_2}, \tag{81}$$

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and further

$$d = \frac{b_2 a_1 (3b_1^2 - 16a_1 c_1)}{2a_2 b_1^2}, \quad f_2 B = -6a_1 - \frac{b_2 a_1^2 (3b_1^2 - 16a_1 c_1)}{a_2 b_1^3}. \tag{82}$$

In the special case of $f_2 = 0$, this solution continues to exist provided $f = -3b_1$ and hence $(16a_1 c_1 - 3b_1^2)a_1 b_2 = 6a_2 b_1^3$.

4. On the other hand, the solution

$$\phi = A\sqrt{1 \pm \tanh(Dx + x_0)}, \quad \psi = F[1 \mp \tanh(Dx + x_0)], \tag{83}$$

is an exact solution to field Eqs. (70) provided

$$D^2 + a_1 = b_1 A^2, \quad 2D^2 + a_1 = 4c_1 A^4, \quad D^2 - a_1 = 4dB^2,$$

$$8D^2 + a_2 = 2f_2 B, \quad 4D^2 - a_2 = 2dA^2, \quad 2D^2 = b_2 B^2. \tag{84}$$

These relations imply that $4c_1 A^2 = b_1 \pm \sqrt{b_1^2 - 4a_1 c_1}$. In the special case of $f_2 = 0$, this solution exists provided $a_2 = -8D^2, dA^2 = 6D^2$ while the other four relations are as given by Eq. (84).

4.2 Solution II

We now present three solutions which are only valid at $m = 1$, i.e. in the hyperbolic limit. For example, it is easy to show that

$$\phi = \frac{A\mathrm{sn}(Dx + x_0, m)}{\sqrt{1 - F\mathrm{sn}^2(Dx + x_0, m)}}, \quad \psi = \frac{B\mathrm{cn}^2(Dx + x_0, m)}{[1 - F\mathrm{sn}^2(Dx + x_0, m)]}, \tag{85}$$

is an exact solution to the coupled Eqs. (70) only if $m = 1$, and if

$$a_1 + dB^2 = (3F - 2)D^2, \quad b_1 A^2 = 2(1 - F)(D^2 + a_1), \quad c_1 A^4 = (1 - F)^2(2D^2 + a_1),$$

$$a_2 + f_2 B = -2(1 + 3F)D^2, \quad dA^2 = (1 - F)[6(1 + F)D^2 - f_2 B], \quad b_2 B^2 = 8D^2 F. \tag{86}$$

From here it follows that $(b_1^2 - 4a_1 c_1)A^4 = 4(1 - F)^2 D^4 > 0$. Note that at $m = 1$, the solution (85) can be rewritten as

$$\phi = \frac{A\tanh(Dx + x_0)}{\sqrt{1 - F\tanh^2(Dx + x_0)}}, \quad \psi = \frac{B\mathrm{sech}^2(Dx + x_0)}{[1 - F\tanh^2(Dx + x_0)]}, \tag{87}$$

Note also that this solution continues to hold good even if $f_2 = 0$.

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4.3 Solution III

Another solution, which is only valid at $m = 1$ is given by

$$\phi = \frac{A\operatorname{sech}(Dx + x_0)}{\sqrt{1 - F\tanh^2(Dx + x_0)}}, \quad \psi = \frac{B\operatorname{sech}^2(Dx + x_0)}{[1 - F\tanh^2(Dx + x_0)]}. \tag{88}$$

This is an exact solution to the coupled Eqs. (70) provided

$$a_1 = D^2, \quad a_2 = 4a_1, \quad b_1 A^2 = 2(1 + F)D^2, \quad b_2 B^2 = 8FD^2,$$

$$c_1 A^4 + dB^2 = 3FD^2, \quad dA^2 + f_2 B = -6(1 + F)D^2. \tag{89}$$

Note that this solution continues to hold good even if $f_2 = 0$.

4.4 Solution IV

Yet another solution, which is only valid at $m = 1$ is given by

$$\phi = \frac{A}{\sqrt{1 - F\tanh^2(Dx + x_0)}}, \quad \psi = \frac{B\operatorname{sech}^2(Dx + x_0)}{[1 - F\tanh^2(Dx + x_0)]}. \tag{90}$$

This is an exact solution to the coupled Eqs. (70) provided

$$b_1 A^2 = 2(1 - F)(D^2 + a_1), \quad dB^2 + a_1 F^2 = (3 - 2F)FD^2, \quad c_1 A^4 = (2D^2 + a_1)(1 - F)^2,$$

$$a_2 F + f_2 B = -2(3 + F)D^2, \quad b_2 B^2 = 8FD^2, \quad dA^2 = (1 - F)(4D^2 - a_2). \tag{91}$$

From here it follows that $(b_1^2 - 4a_1 c_1)A^4 = 4(1 - F)^2 D^4 > 0$. Note that this solution continues to hold good even if $f_2 = 0$.

It is worth noting that for the three solutions as given by Eqs. (87), (88) and (90), while ϕ continues to be a solution of the uncoupled ϕ^6 field theory, in neither of these three cases, ψ is an exact solution of either the symmetric or the asymmetric, uncoupled ψ^4 problem.

5 Conclusions

In this paper we have shown that the Lamé polynomials of order one and two are periodic solutions of a coupled ϕ^6 problem. These are novel solutions in the sense that while they are the solutions of the coupled

ϕ^6 problem, they are not the solutions of the corresponding uncoupled problems. In particular, we have obtained six solutions in terms of Lamé polynomials of order one and two solutions in terms of Lamé polynomials of order two. These results are applicable to both the structural phase transitions [1, 2] and field theoretic contexts [3, 4, 5].

We have also obtained four solutions of the coupled $\phi^6 - \phi^4$ problem, both when the ϕ^4 potential corresponds to a first order as well as a second order transition. Note that while the solutions of the coupled problem are also the solutions of the uncoupled ϕ^6 problem, but they are not the solutions of either the symmetric or the asymmetric uncoupled ϕ^4 problems. These solutions are also useful in understanding coexistence of different crystalline structures in elements [12, 13, 14] and ferroelectrics [15, 16].

It will be interesting to obtain solutions of few other coupled field theories and with couplings that are not bi-quadratic. An example of a coupled model with linear-quadratic coupling occurs in the context of isostructural transitions [20]. It is conceivable that in some cases a linear-cubic coupling may be symmetry allowed.

6   Acknowledgment

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